JOURNAL OF FUNCTIONAL ANALYSIS 7, 297–322 (1968)

A Class of Nonlinear Eigenvalue Problems

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Communicated by Tosio Kato

Received August 4, 1967

1. INTRODUCTION

Let $T(\lambda)$ be a map from the complex numbers $\mathbf{C}$ to the space of linear operators having domain and range in vector space X. By a nonlinear eigenvalue problem we mean the problem of finding an "eigenvalue" $\lambda \in \mathbf{C}$ and a nonzero "eigenvector" $x \in X$, satisfying $T(\lambda)x = 0$. A closely-related problem is that of solving the equation $T(\lambda)x = y$, for a given y. The literature on the problems is considerable, both in finite and infinite-dimensional spaces. Two good bibliographical sources are a 1959 paper by P. H. Müller [*1*] and a recent book by P. Lancaster [*2*]. The former is concerned with the problem in Hilbert space while the latter is devoted exclusively to finite-dimensional spaces. Additional recent references [*3*]–[*16*] have been added at the end of this paper. In what follows, a reference to previous work with no accompanying number will mean that it can be found in [*1*] or [*2*].

As is well known, a normal mode analysis of a vibrating mechanical or electrical system gives rise to an eigenvalue problem for the oscillation frequency λ. Ordinarily only a term λ^2 appears and is treated as a problem linear in the parameter $\mu = \lambda^2$. If there is damping in the system proportional to a first order time derivative, then λ and λ^2 appear. A paper on vibration with damping was published in 1914 by O. Faber who made a fairly complete study of the existence and asymptotic behavior of eigenvalues and eigenfunctions, the Green's function, and expansion properties. In general, with a quadratic dependence on λ, one may expect to get two "complete sets of eigenfunctions", though Faber makes no mention of this.

* Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No. DA-31-124-ARO-D-462 and NSF Grant GP-5574.

C. Miranda, starting in 1936, and D. F. Harazov, beginning in 1945, both studied rational $T(\lambda)$, a typical form being

$$I - \lambda A - \lambda^2 B + \sum_{k=1}^{n} \frac{\lambda^2}{\lambda - a_k} H_k$$

with all operators symmetric; A, B, Hilbert–Schmidt; and $H_1, \ldots, H_n$, finite rank. Miranda examined integral equations and Harazov, both integral equations and operators in Hilbert space. Both authors obtained results paralleling the classical results for linear eigenvalue problems: existence and reality of eigenvalues, successive extremal characterization, expansion theorems, and solvability of inhomogeneous problems. They dealt directly with the nonlinear problems, though they implicitly used some linearization. By linearization we mean the technique (similar to that of writing a single nth order differential equation as a system) of introducing a system of operator equations, in which form the eigenvalue problem becomes linear in λ. This technique appears to have been known for some time to those working with matrices (cf. F. L. Bauer, 1956) and was developed independently by P. H. Müller as an approach to the work of Miranda and Harazov.

In 1955, R. J. Duffin considered "overdamped systems" which lead to quadratic, matrix eigenvalue problems of the form

$$A\lambda^2 + B\lambda + C$$

with A, B, C symmetric; A, B positive; C nonnegative; and $(Bx, x)^2 - 4(Ax, x)(Cx, x) > 0$. He showed that the eigenvalues could be characterized as "minimax" values of a generalized Rayliegh quotient. Duffin's work was extended to more general matrix problems by E. H. Rogers [*13*]. In the paper [*16*] we considered a nonlinear eigenvalue problem arising in the theory of hydrodynamic stability. A preliminary reduction brought the problem to the form

$$Ax - \lambda^2 Bx - \lambda x = 0$$

for positive compact operators A and B in a Hilbert space. The results in [*16*] are similar in nature to those of Duffin and Rogers, though we were unaware of their work at the time and used different methods. The hydrodynamical problem can be put in the form of a linear symmetrizable system, a form generally best suited to handling initial value problems. On the other hand, some of the estimates on eigenvalues obtainable from the nonlinear variational principles appear unavailable from the linear system. We hope to treat the hydrodynamic application, using the two formulations and their interplay, in a separate paper.

In this paper we examine the problem

$$(A - B(\lambda))\, x = 0 \tag{1.1}$$

in a Hilbert space $\mathfrak{H}$, where A is a nonnegative compact operator and $B(\lambda)$ is a polynomial in λ having nonnegative operator coefficients and satisfying $B(0) = 0$. Many of the results can be carried over to operators $B(\lambda)$, which are convergent power series in λ, in a neighborhood of zero. However, for simplicity, we restrict ourselves to polynomials. We will be concerned with the spectrum (defined in Section 2) on the nonnegative real axis. In Section 2 we show that it is discrete and that the corresponding eigenvectors form an unconditional basis for $\mathfrak{H}$ if $A \in C_r$ with $r < 1/2$ (that is, the eigenvalues α_i satisfy $\Sigma \alpha_i^r < \infty$) or if $A \in C_r$ with $r < 2/3$ and $B(\lambda) \in C_2$. The existence of a basis was shown by M. Shinbrot [*14*] for a problem $Ax - \lambda^\alpha B(\lambda)x - \lambda x = 0$ with $a > 1$ and $B(\lambda)$ a bounded linear operator, Lipschitz continuous in λ. He requires a smallness condition on $A = A^*$ or $B(\lambda)$ and, letting the spectrum of A consist of eigenvalues μ_n having nearest neighbor distance δ_n, assumes that

$$\sum | \mu_n{}^\alpha \delta_n^{-1} |^2 \tag{1.2}$$

is small. The methods of Section 2 can be applied to (1.1) under the assumption that (1.2) converges, but we do not do so in this paper. The assumption (1.2) is reasonable for inverses of ordinary differential operators, but is not well suited to partial differential operators where one has no information about δ_n. In that case, a C_p class condition may still be verified. Further, as happens in the hydrodynamic example, information on eigenvalue spacing may be lost in reducing a problem to a convenient form.

In Section 3 we show that all of the nonnegative eigenvalues of $A - B(\lambda)$ can be characterized by variational principles if the eigenvectors form a basis; in particular, if the conditions of Section 2 are satisfied. With no assumptions on the eigenvectors, we show that some of the eigenvalues can still be obtained from variational principles and thus estimated, using monotonicity theorems. We point out that the spectrum, exclusive of that on the nonnegative real axis, may be real or complex and thus a linear approach would yield no variational principles from existing theory. Looked at the other way, we obtain variational principles for the ordinary spectrum of a class of nonselfadjoint operators.

2. Eigenvectors

For notational convenience we will work in a separable Hilbert space $\mathfrak{H}$. We consider $T(\lambda)$ having the form $A - B(\lambda)$, $B(\lambda) \equiv \sum_{k=1}^{N} \lambda^k B_k$ where A is compact and all operators involved are bounded and nonnegative. We call an operator B nonnegative if $(Bx, x) \geqslant 0$ for all $x \in \mathfrak{H}$, and call B positive if $(Bx, x) > 0$ for $x \neq 0$. We will assume that B_1 is positive and that (1.1) is equivalent to a problem of the same form with $B_1 = I$. This will certainly be the case if $(B_1 x, x) \geqslant b_1(x, x)$ for some $b_1 > 0$ and can be shown to be the case when B_1^{-1} is unbounded under conditions on A and the operators B_k relative to B (see [*16*], Section 2). Henceforth, when referring to (1.1) we will mean the problem with $B_1 = 1$. Having set $B_1 = I$, we will use the notation $B_1(\lambda)$ for $\sum_{k=2}^{N} \lambda^k B_k$. We further suppose that the operator norm of A, denoted $\| A \|$, is equal to one. This can be achieved by replacing A by $\| A \|^{-1} A$ and B_k by $\| A \|^{-k} B_k$, giving rise to a new problem having the same eigenvectors as the old, with eigenvalues multiplied by $\| A \|^{-1}$.

For a fixed nonzero $x \in \mathfrak{H}$ the expression $(B(\lambda)x, x)$ increases with λ for $\lambda \geqslant 0$ and thus there is a unique nonnegative zero $\lambda = Z(x) \leqslant \| A \| = 1$ for the polynomial $(Ax, x) - (B(\lambda)x, x)$. The functional $Z(x)$ is homogeneous in x and hence can be considered as a function on the unit sphere in $\mathfrak{H}$. Its value can be computed approximately, using Newton's method. Since $(B(\lambda)x, x)$ has all its derivatives positive for $\lambda \geqslant 0$, the Newton interates $\{z_k\}$, starting with $z_0 = 1$, will decrease to $Z(x)$. A sequence of lower bounds $\{w_k\}$ increasing to $Z(x)$ can be obtained by intersecting secants to the curve $y = (B(\lambda)x, x)$ with the line $y = (Ax, x)$. Intersecting the line $y = \lambda(B(1)x, x)$ with $y = (Ax, x)$ we obtain a point with abscissa w_2. Then intersecting the straight line from $(w_1, (B(w_1)x, x))$ to $(z_1, (B(z_1)x, x))$ with $y = (Ax, x)$ gives w_2, etc. It is easily seen that the w_k will increase to $Z(x)$.

Lemma 2.1. *$Z(x)$ is uniformly strongly continuous on the unit sphere in $\mathfrak{H}$.*

Proof. Given $\eta > 0$ and any unit vector x_0, let x be a unit vector satisfying $\| x - x_0 \| < \frac{1}{4}(1 + \| B_1(1) \|)^{-1}\eta$. Let $\lambda = \lambda_0 - \eta$, where $\lambda_0 = Z(x_0)$. Then

$$
\begin{aligned}
&(Ax, x) - (B(\lambda)\, x, x) \\
&\quad = (Ax, x) - (B_1(\lambda)\, x, x) - \lambda(x, x) - (Ax_0, x_0) \\
&\qquad + (B_1(\lambda_0)\, x_0, x_0) + \lambda(x_0, x_0)
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \eta + ((B_1(\lambda_0) - B_1(\lambda))\, x, x) - |(B_1(\lambda_0)\, x, x - x_0)| \\
&\quad - |(B_1(\lambda_0)\, x_0\,, x - x_0)| - |(Ax, x - x_0)| - |(Ax_0\,, x - x_0)| \\
&\geqslant \eta - 2\,\| x - x_0 \| \cdot (1 + \| B_1(1)\|) \\
&\geqslant \eta/2.
\end{aligned}
$$

Similarly, for $\lambda = \lambda_0 + \eta$, $(Ax, x) = (B(\lambda)x, x) \leqslant -\eta/2$, which means that the zero $Z(x)$ must occur for $|\lambda - \lambda_0| < \eta$.

DEFINITION. The set ρ of complex numbers λ, for which $(A - B(\lambda))^{-1}$ exists as a bounded operator on all of $\mathfrak{H}$, we call the resolvent set of $A - B(\lambda)$. The complementary set σ we call the spectrum. If $A - B(\lambda)$ has a nontrivial nullspace, we call λ an eigenvalue.

LEMMA 2.2. *There exists a $\delta > 0$ such that those complex numbers within distance δ of the nonnegative axis $\mathscr{R}^+$, but not on $\mathscr{R}^+$ are in ρ.*

Proof. Let

$$W_1 = \{\lambda \mid \lambda = re^{i\theta}, \quad 0 < |\theta| \leqslant \pi/2N, \quad r > 0\}$$

and

$$W_2 = \{\lambda \mid \lambda = re^{i\theta}, \quad \pi/2N < |\theta| \leqslant \pi/2, \quad 0 < r < r_0\},$$

where $\sum_{k=2}^{N} r_0^{k-1} \| B_k \| < \frac{1}{2} \sin \pi/2N$. If $\lambda \in W_1$,

$$|\operatorname{Im}\{(Ax, x) - (B_1(\lambda)\, x, x) - \lambda(x, x)\}| \geqslant (x, x)\, r \sin |\theta|$$

and if $\lambda \in W_2$, $\lambda((x, x) + \sum_{k=2}^{N} \lambda^{k-1}(B_k x, x)) \geqslant 0$ is impossible for $x \neq 0$. In either case then, λ cannot be an eigenvalue. If $\lambda \in W_1$, $\operatorname{Re}(\lambda^{-1} B_1(\lambda)x, x) \geqslant 0$ and if $\lambda \in W_2$, $\| \lambda^{-1} B_1(\lambda)\| < 1$. In either case, $\tilde{B} = I + \lambda^{-1} B_1(\lambda)$ has a bounded inverse and since A is compact, $A - B(\lambda) = (A\tilde{B}^{-1} - \lambda I)\tilde{B}$ has closed range. Since for $\lambda \in W_1 \cup W_2$, $A - B(\lambda)$ has a trivial nullspace, its range is all of $\mathscr{H}$ and it has a bounded inverse by the closed graph theorem.

DEFINITION. Let $\Delta(\lambda) \equiv dB/d\lambda$ and

$$
\Delta(\lambda_1\,, \lambda_2) = \begin{cases} \dfrac{B(\lambda_1) - B(\lambda_2)}{\lambda_1 - \lambda_2} & \text{for} \quad \lambda_1 \neq \lambda_2 \\ \Delta(\lambda_1) & \text{for} \quad \lambda_1 = \lambda_2\,. \end{cases}
$$

Also, denote $\Delta(\lambda_1\,, \lambda_2) - I$ by $\Delta_1(\lambda_1\,, \lambda_2)$ and $\Delta(\lambda) - I$ by $\Delta_1(\lambda)$.

DEFINITION. For $x, y \in \mathfrak{H}$, let

$$[x, y] = (\Delta(Z(x), Z(y)\, x, y).$$

LEMMA 2.3. *If B is nonnegative and $x^k(k = 1, 2, ...)$* is a sequence of vectors converging weakly to x_0, then

$$(Bx_0, x_0) \leqslant \liminf(Bx^k, x^k). \tag{2.1}$$

Proof. Since $|(Bx, y)| \leqslant (Bx, x)^{1/2}(By, y)^{1/2}$ the result follows as for the case $B = I$.

LEMMA 2.4. *Points in $(\mathscr{R}^+ - \{0\}) \cap \sigma$ are eigenvalues. There are at most countably many with zero as their only possible accumulation point. To each eigenvalue $\lambda > 0$ there are at most a finite number of linearly independent eigenvectors and if x and y are eigenvectors for distinct eigenvalues,* $[x, y] = 0$.

Proof. If $\lambda > 0$ is in σ, then since $A - B(\lambda)$ has closed range, λ must be an eigenvalue. The remainder of the proof follows the lines of Lemma 3.1 of [*16*]. The compactness used in [*16*] can be avoided by using inequality (2.1). Thus, if y_i is a sequence of normalized eigenvectors corresponding to distinct eigenvalues $\lambda_i > 0$, converging to λ_0, y_i will converge weakly to zero and $0 \leqslant \lambda_0 = \limsup \lambda_i = \limsup((Ay_i, y_i) - (B(\lambda_i)y_i, y_i)) \leqslant 0$.

LEMMA 2.5. *Let $y_1, y_2, ..., y_n$ be any set of nonzero eigenvectors corresponding to distinct eigenvalues $\mu_1 > \mu_2 > \cdots > \mu_n \geqslant 0$,* respectively. Then the vectors are linearly independent and

$$\mu_1 > Z(y_1 + y_2 + \cdots + y_n) > \mu_n. \tag{2.2}$$

Proof. For a quadratic $B(\lambda)$ the result is contained in the treatment of the variational principles in [*16*]. For any polynomial $B(\lambda)$ a direct proof can be given using an idea of Duffin (1955, Lemma 4). Since $Z(y)$ is homogeneous of degree zero and $\mu_1 \neq \mu_2$, it is clear that y_1 and y_2 cannot be colinear. Using the strict monotonicity of $B(\lambda)$ we see that

$$((A - B(\mu_1))(y_1 + y_2), (y_1 + y_2)) = ((A - B(\mu_1))\, y_2, y_2) < 0$$

and thus $Z(y_1 + y_2) < \mu_1$. Similarly, $Z(y_1 + y_2) > \mu_2$. Making the inductive hypothesis that the lemma is true for $n - 1$ distinct eigenvalues and letting $y = y_1 + y_2 + \cdots + y_n$, we see that

$$((A - B(\mu_1))\, y, y) = ((A - B(\mu_1))(y_2 + \cdots + y_n), (y_2 + \cdots + y_n)) < 0$$

since $Z(y_2 + \cdots + y_n) < \mu_2 < \mu_1$. We see that y_1 and $y_2 + \cdots + y_n$ are not colinear which, since we've chosen any eigenvectors for the μ_i, means $y_1, y_2, \ldots, y_n$ are linearly independent. Further, $Z(y) < \mu_1$ and a similar argument shows $\mu_n < Z(y)$.

LEMMA 2.6. *The* nth *largest positive eigenvalue of* $A - tB_1(\lambda) - \lambda$, *including multiplicity, is a continuous nonincreasing function of* t *for* $t \geqslant 0$.

Proof. As in the proof of Lemma 2.4, one can show that for fixed $\mu \geqslant 0$ and $t \geqslant 0$, $A - tB_1(\mu)$ has discrete spectrum on $\mathscr{R}^+ - \{0\}$. Let $\lambda_n(t, \mu)$ denote the nth largest eigenvalue, including multiplicity. From the minimax principle for linear problems it follows that $\lambda_n(t, \mu)$ is a continuous, nonincreasing function of $t \geqslant 0$ and $\mu \geqslant 0$. Since, as observed by Shinbrot [*14*], the eigenvalues of $A - tB_1(\lambda) - \lambda$ are just the fixed points of $\lambda_n(t, \mu) = \mu$ or the intersections of the graphs $\lambda = \lambda_n(t, \mu)$ and $\lambda = \mu$, it is easily seen that $\lambda_n(t)$ is continuous and nonincreasing.

The Cauchy integral approach to obtaining projections in linear eigenvalue problems has proved a valuable tool (cf. [*17*]), and we wish to extend its usefulness to the nonlinear setting. We let $R_\lambda \equiv (A - B(\lambda))^{-1}$ for $\lambda \in \rho$. The proofs of the following three results are straightforward.

LEMMA 2.7. *The set* ρ *is open, and for* μ *and* λ *in* ρ *the generalized resolvent equation*

$$R_\mu - R_\lambda = (\mu - \lambda) R_\mu \Delta(\mu, \lambda) R_\lambda \tag{2.3}$$

is valid.

COROLLARY 2.8. *For* μ *and* λ *in* ρ

$$R_\mu - R_\lambda = (\mu - \lambda) R_\mu[\Delta(\mu) + (\lambda - \mu)\tilde{\Delta}] R_\lambda, \tag{2.4}$$

where

$$\tilde{\Delta} \equiv \tilde{\Delta}(\mu, \lambda) = \sum_{k=2}^{N} B_k \sum_{j=1}^{k-1} \mu^{k-1-j} \sum_{l=0}^{j-1} \lambda^{j-1-l}\mu^l.$$

COROLLARY 2.9. R_λ *is analytic in* ρ *and*

$$\frac{d}{d\lambda} R_\lambda = R_\lambda \Delta(\lambda) R_\lambda.$$

LEMMA 2.10. *If $\lambda_0 > 0$ is an eigenvalue of $A - B(\lambda)$, then it is a simple pole of R_λ.*

Proof. The linear operator $A - B_1(\lambda_0)$ has an ordinary eigenvalue at λ_0 and, repeating previous arguments, one sees that it is isolated and has finite multiplicity. Since $A - B_1(\lambda_0)$ is selfadjoint, λ_0 is a simple pole of $R_\lambda^0 = (A - B_1(\lambda_0) - \lambda)^{-1}$. But for $\lambda \in \rho$ near λ_0,

$$\begin{aligned} R_\lambda &= (A - B_1(\lambda) - \lambda)^{-1} \\ &= (A - B_1(\lambda_0) - \lambda + B_1(\lambda_0) - B_1(\lambda))^{-1} \\ &= [(I + (B_1(\lambda_0) - B_1(\lambda))\, R_\lambda^0)(A - B_1(\lambda_0) - \lambda)]^{-1} \\ &= R_\lambda^0[I + (B_1(\lambda_0) - B_1(\lambda))\, R_\lambda^0]^{-1}. \end{aligned}$$

If $H(\lambda) \equiv I + (B_1(\lambda_0) - B_1(\lambda))R_\lambda^0$ has an inverse for λ in a full neighborhood of λ_0, then R_λ will have a simple pole at λ_0. Near λ_0

$$R_\lambda^0 = \frac{P(\lambda_0)}{\lambda_0 - \lambda} + Q(\lambda), \tag{2.5}$$

where $P(\lambda_0)$ is the perpendicular projection associated with $A - B_1(\lambda_0)$ at the ordinary eigenvalue λ_0, and $Q(\lambda)$ is analytic in a neighborhood of λ_0. If we show that $H(\lambda_0) = I + \Delta(\lambda_0)P(\lambda_0)$ has an inverse, we can conclude that $H(\lambda)$ is invertible in a full neighborhood of λ_0. Since $\Delta(\lambda_0)P(\lambda_0)$ is compact, if $H(\lambda_0)$ is not invertible, $H(\lambda_0)y = 0$ for some $y \neq 0$. But then

$$0 = (P(\lambda_0)\, y, H(\lambda_0)\, y) = (P(\lambda_0)\, y, (I + \Delta(\lambda_0))\, P(\lambda_0)\, y)$$

which, since $\Delta(\lambda_0) \geqslant 0$, implies $P(\lambda_0)y = 0$ and finally, $y = 0$. The contradiction shows that R_λ has only simple poles on the positive real axis.

LEMMA 2.11. *Let λ_0 be an eigenvalue of $A - B(\lambda)$ and define*

$$F(\lambda_0) = \frac{-1}{2\pi i} \oint_{C_{\lambda_0}} R_\lambda \Delta(\lambda)\, d\lambda \tag{2.6}$$

where C_{λ_0} is a closed contour in ρ, surrounding λ_0, but no other eigenvalue. Then $F(\lambda_0)$ is a projection onto the subspace of eigenvectors of $A - B(\lambda)$ for the eigenvalue λ_0.

Proof. Let C_1 and C_2 be circles of radii r and $2r$ respectively, centered at λ_0. Then for r sufficiently small

$$F^2(\lambda_0) = \frac{1}{(2\pi i)^2} \oint_{C_2} R_\mu \Delta(\mu)\, d\mu \cdot \oint_{C_1} R_\lambda \Delta(\lambda)\, d\lambda.$$

Using the generalized resolvent equation we see that

$$\begin{aligned} F^2(\lambda_0) &= \frac{1}{(2\pi i)^2} \oint_{C_2} \oint_{C_1} \frac{R_\mu - R_\lambda}{\mu - \lambda} \Delta(\lambda)\, d\mu\, d\lambda \\ &\quad + \frac{1}{(2\pi i)^2} \oint_{C_2} \oint_{C_1} (\mu - \lambda)\, R_\mu \tilde{\Delta}(\mu, \lambda)\, R_\lambda \Delta(\lambda)\, d\mu\, d\lambda \\ &= \frac{1}{(2\pi i)^2} \oint_{C_2} \oint_{C_1} \frac{R_\mu}{\mu - \lambda} \Delta(\lambda)\, d\mu\, d\lambda \\ &\quad - \frac{1}{(2\pi i)^2} \oint_{C_2} \oint_{C_1} \frac{R_\lambda}{\mu - \lambda} \Delta(\lambda)\, d\mu\, d\lambda + O(r). \end{aligned}$$

The term $O(r)$ results from the fact that R_λ has first-order reciprocal growth near λ_0. Integrating $(\mu - \lambda)^{-1} R_\mu \Delta(\lambda)$ with respect to λ on C_1 gives 0. The last integral above, evaluated first with respect to μ, is just $F(\lambda_0)$, and letting $r \to 0$ we see that $F^2(\lambda_0) = F(\lambda_0)$. It remains to show that $F(\lambda_0)$ and $P(\lambda_0)$ have the same range, or that $F(\lambda_0)P(\lambda_0) = P(\lambda_0)$ and $P(\lambda_0)F(\lambda_0) = F(\lambda_0)$.

Using the notation introduced above we can write

$$\begin{aligned} F(\lambda_0) &= \frac{-1}{2\pi i} \oint_{C_{\lambda_0}} R_\lambda^0 [I + (B_1(\lambda_0) - B_1(\lambda))\, R_\lambda^0]^{-1}\, \Delta(\lambda)\, d\lambda \\ &= -P(\lambda_0)\, H^{-1}(\lambda_0)\, \Delta(\lambda_0) \end{aligned}$$

so that $P(\lambda_0)F(\lambda_0) = F(\lambda_0)$. The expression $(I + (B_1(\lambda_0) - B_1(\lambda))R_\lambda^0)P(\lambda_0)$ is analytic at λ_0 and can be written $(\Delta(\lambda_0) + G(\lambda))P(\lambda_0)$, where $G(\lambda)$ is analytic near zero and vanishes at $\lambda = \lambda_0$. Then

$$\begin{aligned} P(\lambda_0) &= \frac{-1}{2\pi i} \oint_{C_{\lambda_0}} R_\lambda^0 P(\lambda_0)\, d\lambda \\ &= \frac{-1}{2\pi i} \oint_{C_{\lambda_0}} R_\lambda [I + (B_1(\lambda_0) - B_1(\lambda))\, R_\lambda^0]\, P(\lambda_0)\, d\lambda \\ &= \frac{-1}{2\pi i} \oint_{C_{\lambda_0}} R_\lambda (\Delta(\lambda_0) + G(\lambda))\, P(\lambda_0)\, d\lambda \\ &= F(\lambda_0)\, P(\lambda_0). \end{aligned}$$

LEMMA 2.12. *Let P be a projection in $\mathfrak{H}$. Then $(I + P - P^*)^{-1}$ exists and $P(I + P - P^*)^{-1}$ is the orthogonal projection on $\mathscr{R}(P)$, the range of P.*

Proof. Since $i(P - P^*)$ is selfadjoint, the operator $I + P - P^* =$

$I - i[i(P - P^*)]$ is normal and has a numerical range W on the line $\operatorname{Re} \lambda = 1$. The closure of the numerical range contains the spectrum, so $(I + P - P^*)^{-1}$ exists as does the adjoint $(I - P + P^*)^{-1}$. Now suppose $x \in \mathscr{R}(P)$ and let $P(I + P - P^*)^{-1}x = y$. If $(I + P - P^*)^{-1}x = z$, then $Pz = y$ and $P^*x = P^*(I + P - P^*)z = P^*y$. Noting that $(I - P)x = (I - P)y = 0$ and adding, one finds $(I - P + P^*)x$ $(I - P + P^*)y$ or $x = y$. If x is perpendicular to $\mathscr{R}(P)$, then $P^*x = 0$. With $P(I + P - P^*)^{-1}x = w$ and $(I + P - P^*)^{-1}x = v$, we obtain $P^*w = P^*Pv = P^*x = 0$. Adding $(I - P)w = 0$ yields $(I - P + P^*)w = 0$ or $w = 0$. It follows that $P(I + P - P^*)^{-1}$ is the desired projection.

LEMMA 2.13. *If P and Q are projections in $\mathfrak{H}$ with $\| P - Q \| < 1$, then with $P_\perp = P(I + P - P^*)^{-1}$ and $Q_\perp = Q(I + Q - Q^*)^{-1}$, the operator*

$$U = P_\perp(I + P_\perp(Q_\perp - P_\perp) P_\perp)^{-1/2} Q_\perp \tag{2.7}$$

is a partial isometry with $\mathscr{R}(Q)$ as its initial domain and $\mathscr{R}(P)$ as its final domain.

Proof. The lemma will follow from Lemma 2.12 and the corresponding result for selfadjoint projections (cf. [*18*], p. 268) if we show that $\| P_\perp - Q_\perp \| < 1$. We have

$$\begin{aligned} \| Qx - Px \| &= \| Q_\perp x + Q(I - Q_\perp x) - P_\perp x - P(I - P_\perp) x \| \\ &= \|(I - Q)(Q_\perp - I) x + (I - P_\perp) x - P(I - P_\perp) x \| \\ &\geqslant \|(I - P_\perp) x - P_\perp P(I - P_\perp) x \| - \|(I - Q)(Q_\perp - I) x \| \\ &\geqslant \|(I - P_\perp) x \| - \|(I - Q)(Q_\perp - I) x \|. \end{aligned} \tag{2.8}$$

Since

$$\| P_\perp - Q_\perp \| = \sup_{\|x\|=1} |((P_\perp - Q_\perp) x, x)| = \sup_{\|x\|=1} | \, \| P_\perp x \|^2 - \| Q_\perp x \|^2 \, |,$$

it is clear that $\| P_\perp - Q_\perp \| \leqslant 1$. If $\| P_\perp - Q_\perp \| = 1$, then there exists a sequence of unit vectors x^k such that $\| P_\perp x^k \| \to 0$ and $\| Q_\perp x^k \| \to 1$, or such that the limits exist with 0 and 1 interchanged. Using symmetry we may assume the first alternative, or that $P_\perp x^k \to 0$ and $(I - Q_\perp)x^k \to 0$. However, letting $x = x^k$ in (2.8) and taking the limit as $k \to \infty$, we must conclude that $\| P - Q \| \geqslant 1$, which is a contradiction.

As a corollary we obtain the following well-known result:

COROLLARY 2.14. *If P and Q are projections with $\| P - Q \| < 1$, then $\mathscr{R}(P)$ and $\mathscr{R}(Q)$ have the same dimension.*

DEFINITION. We say that a collection of projections $\{P_\alpha\}$, for α in an index set A, is disjoint if $P_{\alpha_1}P_{\alpha_2} = 0$ when $\alpha_1 \neq \alpha_2$.

DEFINITION. If $Q_1, \ldots, Q_m$ is a set of projections, we say that the ranges $\mathscr{R}(Q_i)$ are linearly independent if, whenever $Q_i x_i = x_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m x_i = 0$, then $x_i = 0$ for all i.

LEMMA 2.15. *Let $Q_1, Q_2, \ldots, Q_m$ be a collection of projections with linearly independent ranges. If $S = \sum_{i=1}^m Q_i$ has a bounded inverse, then the collection of operators*

$$\tilde{Q}_i = Q_i S^{-1}, \qquad 1 \leqslant i \leqslant m,$$

is a set of disjoint projections with $\mathscr{R}(\tilde{Q}_i) = \mathscr{R}(Q_i)$ for $1 \leqslant i \leqslant m$ and $\sum_{i=1}^m \tilde{Q}_i = I$.

Proof. Clearly $\Sigma \tilde{Q}_i = I$ and thus if $x \in \mathscr{R}(Q_j)$,

$$0 = x - \sum_{i=1}^{m} \tilde{Q}_i x = x - Q_j S^{-1} x - \sum_{i \neq j} Q_i S^{-1} x.$$

Using the linear independence, we conclude that $x = Q_j S^{-1} x = \tilde{Q}_j x$ and $\tilde{Q}_i x = 0$ for $i \neq j$.

In what follows we will use the symbol $O(s)$, $s > 0$, for a complex number or a linear operator, according to context, with the following understanding: There is a constant $C > 0$ such that

(1) the complex number $O(s)$ satisfies $|O(s)| \leqslant C \cdot s$,
(2) the operator $O(s)$ satisfies $\|O(s)\| \leqslant C \cdot s$.

The constant C will vary from one line to another and can be computed recursively. However, it never exceeds a fixed absolute constant $\tilde{C}$.

Suppose that the operator A appearing in (1.1) is in class C_r $(r > 0)$ (cf. [*19*], p. 1088); that is, if $\alpha_i \geqslant 0$ are the eigenvalues of A, including multiplicity, $\sum_{i=1}^\infty \alpha_i^r < \infty$. For convenience we will assume, by further normalization if necessary, that $\sum \alpha_i^r = 1$. As a consequence, there can be at most a^{-r} eigenvalues of A on the interval $[a, 1]$ for any $a > 0$. Further, in an interval $[a, b]$ with $0 < a < b \leqslant 1$, there must be a subinterval, of length at least $(b - a)(1 + a^{-r})^{-1}$, containing no eigenvalue. Letting $a = 1/({}_2k + 1)$ and $b = 1/({}_2k)$ for $k = 1, 2, 3, \ldots$, we see that in the interval $[1/({}_2k+1), 1/({}_2k)]$ there must be a gap of length at least $g_k = \frac{1}{2} \cdot 2^{-(k+1)(r+1)}$ in the spectrum of A. Let m_k be the midpoint of the gap.

Letting $A_\lambda = (A - \lambda)^{-1}$ and $R_\lambda(t) = (A - tB_1(\lambda) - \lambda)^{-1}$, we see that

$$R_\lambda(t) = A_\lambda(I - tB_1(\lambda)\, A_\lambda)^{-1}$$

when $(I - tB_1(\lambda)A_\lambda)^{-1}$ exists. Suppose we let $b = \max_{2\leqslant j\leqslant N}\| B_j \|$. Then for $\lambda = m_k$,

$$\| tB_1(m_k)\, A_{m_k} \| \leqslant t \sum_{j=2}^{N} m_k^{\,j} \| B_j \| \cdot 2g_k^{-1} = O(tb2^{k(r-1)}).$$

If $O(b2^{k_0(r-1)}) < 1$, then for $k \geqslant k_0$ and $0 \leqslant t \leqslant 1$ the point m_k will be in the resolvent set of $A - tB_1(\lambda) - \lambda$. It follows from Lemma 2.6 and the fact that $A \in C_r$, that the number of eigenvalues of $A - tB_1(\lambda) - \lambda$ above m_{k_0} is independent of $t \in [0, 1]$ and is at most $m_{k_0}^{-r}$, while the number between m_k and m_{k+1} for $k \geqslant k_0$ is also constant and at most m_{k+1}^{-r}. Let the eigenvalues α_i of A between 1 and m_{k_0} be numbered with subscripts $1, 2,\ldots, n_{k_0}$ including multiplicity; those between m_{k_0} and m_{k_0+1}, numbered $n_{k_0}+1, n_{k_0}+2,\ldots, n_{k_0+1}$, and so on. Denote the index set $n_k + 1, n_k + 2,\ldots, n_{k+1}$ by J_k. Inasmuch as $A - B(\lambda)$ has the same number of eigenvalues between gaps as does A, the eigenvalues λ_i of $A - B(\lambda)$ can be numbered using the same sets J_k. With each eigenvalue λ_i of $A - B(\lambda)$ we associate a rank one projection F_i. For an eigenvalue of multiplicity one, F_i is obtained from (2.6). If λ is an eigenvalue of higher multiplicity, we may choose projections F_i with orthogonal ranges for those $\lambda_i = \lambda$, provided, of course, that they sum to $F(\lambda)$ given by (2.6). Similar remarks hold for the projections associated with A which we denote by E_i. The eigenspace for $A - B(\lambda)$ at $\lambda = 0$ is just the nullspace of A. If the dimension of $\mathcal{N}(A)$ is $n + 1 \leqslant \infty$, we let J_0 be the set $\{0, -1, -2,\ldots, -n\}$ and extend the indexing to $\lambda_i = \alpha_i = 0$ and $E_i = F_i$ for $i \in J_0$ in an obvious way. Finally, we denote a unit vector in $\mathscr{R}(F_i)$ by x_i, and in $\mathscr{R}(E_i)$ by e_i. The set $\{e_i\}$ is then an orthonormal basis for $\mathfrak{H}$.

We suppose now that $A \in C_r$ with $r < 2/3$ and let $\tilde{r} = r/3 + 1/2$, which is greater than r. Let Γ_k be the rectangle with corners $(m_k \pm i2^{-2\tilde{r}k}, m_{k+1} \pm i2^{-2\tilde{r}k})$. For k large, Γ_k will surround only real spectrum of $A - B(\lambda)$ and we may assume it for all k by shrinking a finite number of the rectangles. If we integrate $(-2\pi i)^{-1}(A_\lambda - R_\lambda\Delta(\lambda))$ counterclockwise around Γ_k, we obtain $\sum_{J_k}(E_i - F_i)$. We have

$$\begin{aligned} A_\lambda - R_\lambda\Delta(\lambda) &= A_\lambda - A_\lambda[I + B_1A_\lambda(I - B_1A_\lambda)^{-1}](I + \Delta_1(\lambda)) \\ &= -A_\lambda B_1(\lambda)\, A_\lambda(I - B_1A_\lambda)^{-1}\,(I + \Delta_1(\lambda)) - A_\lambda\Delta(\lambda), \end{aligned}$$

assuming $(I - B_1(\lambda)A_\lambda)^{-1}$ exists. On Γ_k, $B_1(\lambda) = O(b2^{-2k})$ and $\| A_\lambda \|$ is bounded by the reciprocal of the distance from Γ_k to the spectrum of A, that is, by $O(2^{k(r+1)})$. Then $B_1(\lambda)A_\lambda = O(b2^{k(r-1)})$ and for k large, $I - B_1(\lambda)A_\lambda$ has a bounded inverse on Γ_k. Further, on Γ_k, $\Delta_1(\lambda) = O(b \cdot 2^{-k})$. For k large,

$$\left\| \frac{1}{2\pi i} \oint_{\Gamma_k} A_\lambda B_1 A_\lambda (I - B_1 A_\lambda)^{-1} (I + \Delta_1)\, d\lambda \right\| = O\left(\int_{\Gamma_k} \| A_\lambda B_1 A_\lambda \|\, d\lambda \right). \quad (2.9)$$

One can verify, using the estimates given above, that the integral (2.9) on the sections of Γ_k parallel to the imaginary axis is $O(b2^{2k(r-\tilde{r})})$ and that on the remainder of Γ_k it is $O(b2^{k(4\tilde{r}-3)})$. Since $\tilde{r} = r/3 + 1/2$, (2.9) is $O(b2^{k[4r-3]/3}$. The integral of $A_\lambda \Delta_1$ on Γ_k has an even smaller bound and thus, letting $\sum_{J_k} E_i = E^k$ and $\sum_{J_k} F_i = F^k$, we have the following lemma:

LEMMA 2.16. *For all sufficiently large k*

$$E^k - F^k = O(b2^{k[4r-3]/3}).$$

LEMMA 2.17. *If $\{v_i\}$, $i \in J_k$ is any set of complex numbers, then*

$$(1 - 16b2^{k(r-1)}) \sum_{J_k} (v_i)^2 \leqslant \sum_{J_k} \sum_{J_k} (x_i, x_j)\, v_i \bar{v}_j \leqslant (1 + 16b2^{k(r-1)}) \sum_{J_k} (v_i)^2. \quad (2.10)$$

Proof. First of all, the matrix (x_i, x_j) for $i, j \in J_k$ has at most $2^{(k+2)r}$ rows and columns. The diagonal consists of ones and since $[x_i, x_j] = 0$ for eigenvectors corresponding to distinct eigenvalues, the off-diagonal entries are either zero or equal to

$$(\Delta_1(\lambda_i, \lambda_j)\, x_i, x_j). \quad (2.11)$$

For $i \in J_k$, $| \lambda_i | < 2^{-k}$ and thus $\sum_{i \neq j} |(x_i, x_j)|^2 \leqslant 2^{(k+2)r} \cdot (4b \cdot 2^{-k})^2$. A standard estimate then gives (2.10).

Let $\mathcal{G}_k$ denote the linear span of the eigenvectors x_i for $i \in J_k$, and let I_k denote the identity map in $\mathcal{G}_k$.

LEMMA 2.18. *For all sufficiently large k, there exists a linear map $\tilde{N}_k$ acting in $\mathcal{G}_k$, with $\| \tilde{N}_k - I_k \| = O(b2^{k(r-1)})$, such that the vectors $\tilde{N}_k x_i$ for $i \in J_k$ form an orthonormal basis of $\mathcal{G}_k$.*

Proof. Let w_i, $i \in J_k$, be an orthonormal basis for $\mathcal{G}_k$ and let K be a linear operator mapping w_i to x_i. The map K has a polar decomposition $K = UN$ where $N = (K^*K)^{1/2}$ is positive and U is unitary.

The matrix representing K^*K with respect to the basis w_i has entries $(w_i, K^*Kw_j) = (x_i, x_j)$. From the previous lemma one sees that

$$I_k(1 - 16b2^{k(r-1)}) \leqslant K^*K \leqslant I_k(1 + 16b2^{k(r-1)})$$

and thus $N = (K^*K)^{1/2} = I + V$ with $V = O(b2^{k(r-1)})$. Writing

$$x_i = UNw_i = U(I + V)\, w_i = Uw_i + UVw_i = Uw_i + UVK^{-1}x_i$$

or $(I_k - UVK^{-1})x_i = Uw_i$, we see that Uw_i will serve as the orthonormal basis required if we let $\tilde{N}_k = I_k - UVK^{-1}$. It is easily seen that $UVK^{-1} = O(b2^{k(r-1)})$ in $\mathscr{G}^k$, which completes the proof.

LEMMA 2.19. *Let x be a normalized eigenvector in $\mathscr{G}_k$, and e a unit vector in $\mathscr{R}(I - \sum_{J_k} E_i)$. Then*

$$(x, e) = O((b + b^2)\, 2^{k([3r/2]-1)})$$

for all sufficiently large k.

Proof. Suppose x is an eigenvector for λ_i. Then $P(\lambda_i)x = x$, where $P(\lambda_i)$ is the projection on the full eigenspace associated with $A - B_1(\lambda_i)$ at λ_i. As above one sees that the points m_k and m_{k+1} will be in the resolvent set of $A - B_1(\lambda_i)$ and we can compare the projections associated with A and $A - B_1(\lambda_i)$ for eigenvalues in $[m_{k+1}, m_k]$. We can expand $(A - B_1(\lambda_i) - \lambda)^{-1} \cdot A_\lambda$ as

$$A_\lambda B_1 A_\lambda + A_\lambda B_1 A_\lambda B_1 A_\lambda (I - B_1 A_\lambda)^{-1}$$

and can write

$$A_\lambda = \sum_i \frac{E_i}{\alpha_i - \lambda}$$

in which case

$$\frac{1}{2\pi i} \oint_{\Gamma_k} A_\lambda B_1(\lambda_i)\, A_\lambda\, d\lambda = S + S^*$$

where

$$S = \sum_{l \in J_k} E_l B_1(\lambda_i) \sum_{j \in J_k^c} \frac{E_j}{\alpha_l - \alpha_j},$$

J_k^c indicating the index set complementary to J_k. Since

$$|(Sx, y)| \leqslant \| B_1(\lambda_i)\| \left(\sum_{l \in J_k} \left\| \sum_{j \in J_k^c} \frac{E_j x}{\alpha_l - \alpha_j} \right\|^2\right)^{1/2} \left(\sum_{J_k} \| E_l y \|^2\right)^{1/2}$$
$$\leqslant 8b \cdot 2^{-2k} \cdot (2^{rk} \cdot g_k^{-2})^{1/2} \cdot \| x \| \cdot \| y \|$$

we have $\| S \| = O(b2^{k([3r/2]-1)})$. The integral of $A_\lambda B_1 A_\lambda B_1 A_\lambda (I - B_1 A_\lambda)^{-1}$ around Γ_k is $O(b^2 2^{k([6r/5]-7/5)})$ and thus, letting

$$\frac{1}{2\pi i} \oint_{\Gamma_k} (A - B_i(\lambda_i) - \lambda)^{-1} = Q_i ,$$

we have shown that $E^k - Q_i = O((b + b^2)2^{k([3r/2]-1)})$ for k large and $i \in J_k$. Now

$$\begin{aligned}(e, x) &\doteq ((I - E^k)\, e, Q_i x) = ((I - E^k)\, e, (Q_i - E^k)\, x) \\ &= O((b + b^2)\, 2^{k([3r/2]-1)}).\end{aligned}$$

LEMMA 2.20. *Under the hypotheses of Lemma* 2.19, *suppose that coefficients of $B_1(\lambda)$ are Hilbert–Schmidt operators ($B_k \in C_2$) and that the* 2-*norm*, $\| B_k \|_2 \leqslant \beta$ *for each* $2 \leqslant k \leqslant N$. *Then*

$$(x, e) = O((\beta + b^2)\, 2^{k(r-1)})$$

for all sufficiently large k.

Proof. Referring to the previous proof, we can estimate $|(Sx, y)|$ by

$$\left| \sum_{l \in J_k} \left(\| E_l y \| \cdot \sum_{j \in J_k^c} \frac{E_j x}{\alpha_l - \alpha_j} , B_1(\lambda_i) \frac{E_l y}{\| E_l y \|} \right) \right|$$

omitting terms in which $E_l y = 0$. We have already seen that

$$\left\| \sum_{j \in J_k^c} \frac{E_j x}{\alpha_l - \alpha_j} \right\| \leqslant g_k^{-1} \| x \|$$

and since $\| E_l y \|^{-1} E_l y$ is an orthonormal set applying the Schwarz inequality above yields an upper bound

$$O(\lambda_i^2 g_k^{-1} \| x \| \cdot (\Sigma \| E_l \|^2)^{1/2} \cdot \beta) = O(\beta 2^{k(r-1)}) \| x \| \cdot \| y \|.$$

From this point the proof is the same as the previous one.

LEMMA 2.21. *Suppose $A \in C_r$ for $r < 1/2$ or that $r < 2/3$ and $B(\lambda) \in C_2$. Then for all sufficiently large k, the operators*

$$\hat{F}_i = F_i (I + F^k - E^k)^{-1}$$

and

$$(I - E^k)(I + F^k - E^k)^{-1}$$

are a collection of disjoint projections with ranges $\mathscr{R}(\hat{F}_i) = \mathscr{R}(F_i)$, $i \in J_k$ and $\mathscr{R}(I - E^k)$, respectively.

Proof. Using Lemma 2.15 we need only show that the ranges involved are linearly independent or that a unit vector e in $\mathscr{R}(I - E^k)$ and the eigenvectors x_i, $i \in J_k$, are linearly independent. That will be the case if the matrix

$$\begin{pmatrix} (e, e) & (e, x_{n_k+1}) & \cdots & (e, x_{n_{k+1}}) \\ (x_{n_k+1}, e) & & & \\ \vdots & & & \\ (x_{n_{k+1}}, e) & & & (x_{n_{k+1}}, x_{n_{k+1}}) \end{pmatrix}$$

is nonsingular. The diagonal entries are all one and the sum of the squares of the off-diagonal elements is $O((b + b^2)2^{k(4r-2)})$, or $O((\beta + b^2)2^{k(3r-2)})$ if $B(\lambda) \in C_2$. Then if $r < 1/2$ or $r < 2/3$, respectively, the matrix above will be nonsingular for large k.

We now let $\tilde{F}^k = \sum_{i \in J_k} \tilde{F}_i$ and define an extension of $\tilde{N}_k$, defined in Lemma 2.18, to the whole space by setting

$$N_k = \tilde{N}_k \tilde{F}^k + (I - \tilde{F}^k).$$

Since $I - \tilde{F}^k = (I - E^k)(I + F^k - E^k)^{-1}$ and $F^k - E^k = O(b2^{k[4r-3]/3})$, by Lemma 2.16, it is easy to see that $I - \tilde{F}$ is uniformly bounded in k for large k. Thus $N_k - I = (N_k - I_k)\tilde{F}_k = O(b2^{k(r-1)})$ for large k.

LEMMA 2.22. *The operators*

$$H_i = W_k N_k \tilde{F}_i N_k^{-1} W_k^{-1}, \qquad i \in J_k,$$

where

$$W_k = E^k[I + E^k(F_\perp{}^k - E^k) E^k]^{-1/2} F_\perp{}^k \tilde{F}^k + (I - E^k)(I + F^k - E^k)^{-1}$$

and

$$F_\perp{}^k = \tilde{F}^k(I + \tilde{F}^k - (\tilde{F}^k)^*)^{-1}$$

are a disjoint collection of orthogonal projections satisfying

$$H_i = H_i E^k = E^k H_i$$

and

$$\sum_{i \in J_k} H_i = E^k.$$

Proof. The map W_k takes $\mathscr{R}(\tilde{F}^k)$ isometrically onto $\mathscr{R}(E^k)$ by Lemmas 2.13 and 2.12, and acts as the identity on $\mathscr{R}(I - E^k)$. The operator W_k has an inverse, namely,

$$V_k = F_\perp{}^k[I + E^k(F_\perp{}^k - E^k) E^k]^{-1/2} E^k + I - E^k.$$

If $x \in \mathscr{R}(I - E^k)$, then $H_i x = 0$ for i in J_k. Letting $\theta_i = W_k N_k x_i$ for $i \in J_k$ and recalling Lemma 2.18, it is easy to see that θ_i, $i \in J_k$ constitutes an orthonormal basis for $\mathscr{R}(E^k)$ and that $H_j \theta_i = \delta_{ij} \theta_j$.

Definition. By a spectral measure on the integers we mean a map $P(\cdot)$ from the subsets of the integers Z to the bounded operators on $\mathfrak{H}$ with the following properties: For any subsets $J_1, J_2 \subset Z$

(1) $P(J_1)P(J_2) = P(J_1 \cap J_2)$,

(2) $P(J_1) \cup P(J_2) \equiv P(J_1) + P(J_2) - P(J_1)P(J_2) = P(J_1 \cup J_2)$,

(3) $P(\emptyset) = 0$, $P(Z) = I$, $P(Z - J_1) = I - P(J_1)$,

(4) $\| P(J_1)\| \leqslant K$, K independent of J_1.

By a theorem of Lorch and Mackey (cf. [*20*]) such a measure is similar to one that is selfadjoint. That is, there exists a bounded linear map M with a bounded inverse, such that $M^{-1}P(\cdot)M$ is a self-adjoint spectral measure. In the sequel we further require spectral measures to be strongly countable additive. If for each $i \in Z$, $P(i)$ is a one-dimensional projection and h_i is a unit vector in the range of $P(i)$, a consequence of countable additivity is that each $x \in \mathfrak{H}$ can be uniquely represented as $x = \sum_{i \in Z} a_i h_i$, the series being unconditionally convergent. Such a set $\{h_i\}$ is called an unconditional basis. Conversely, a set $\{h_i\}$ in terms of which each vector can be expanded in an unconditionally convergent series gives rise to a spectral measure (see Lorch [*21*]). We will call a set of vectors an "unconditional set" if it is an unconditional basis for its closed linear span $\mathfrak{H}_0$. Such a set can be extended to an unconditional basis, for example, by adjoining an orthonormal basis for the orthogonal complement of $\mathfrak{H}_0$ in $\mathfrak{H}$.

Lemma 2.23. *Let $P^n(\cdot)$ be a sequence of spectral measures on the integers such that*

(1) $\| P^n(J)\| \leqslant K$, *for some K independent of J and n,*

(2) *Given $\epsilon > 0$, there exists an integer $N(\epsilon)$, such that for any $J \subset Z$,*

$$\| P^n(J) - P^m(J)\| < \epsilon$$

for m and n greater than $N(\epsilon)$.

Then the spectral measures P^n have a limit P which is a spectral measure on the integers.

Proof. The proof is straightforward and is omitted.

THEOREM 2.24. *If $A \in C_r$ for $r < 1/2$ or if $A \in C_r$ for $r < 2/3$ and $B(\lambda) \in C_2$, then the eigenvectors $\{x_i\}$ corresponding to nonnegative eigenvalues of $A - B(\lambda)$ form an unconditional basis for $\mathfrak{H}$.*

Proof. If k_1 is sufficiently large and $k_2 > k_1$, then the ranges of the projections F_i for $i \in J_k$ and $k_1 \leqslant k \leqslant k_2$ are linearly independent. For suppose that $e \in \mathscr{R}(I - \sum_{k=k_1}^{k_2} E^k)$, $y_i \in \mathscr{R}(F_i)$ and $e = \Sigma y_i$, the sum being over $i \in J_k$ with $k_1 \leqslant k \leqslant k_2$. If $e = 0$, then all $y_i = 0$ by the linear independence of finite sets of eigenvectors. If $e \neq 0$, we can asume $\| e \| = 1$. Partitioning the sum, we can write $e = z_k + z_{k_1+1} + \cdots + z_{k_2}$, where $z_k \in \mathscr{G}_k$. Letting $\tilde{z}_k = W_k z_k \in \mathscr{R}(E_k)$ we have $e = \Sigma W_k^{-1} \tilde{z}_k$. Using Lemma 2.16, a short computation shows that $\| W_k^{-1} - I \| = O(b2^{k[4r-3]/3})$. Thus $e = \Sigma \tilde{z}_k + \Sigma (W_k^{-1} - I) \tilde{z}_k$ and since the vectors $\tilde{z}_k$ and e are orthogonal,

$$\left\| \sum_{k=k_1}^{k_2} (W_k^{-1} - I)\, \tilde{z}_k \right\|^2 = 1 + \sum_{k=k_1}^{k_2} \| \tilde{z}_k \|^2.$$

But then

$$1 + \sum_{k=k_1}^{k_2} \| \tilde{z}_k \|^2 \leqslant \left(\sum_{k=k_1}^{k_2} O(b2^{k(4/3[r-1])}) \| \tilde{z}_k \| \right)^2$$

$$\leqslant \sum_{k=k_1}^{k_2} O^2(b2^{k[4r-3]/3}) \cdot \sum_{k=k_1}^{k_2} \| \tilde{z}_k \|^2$$

which, for k_1 large, is impossible.

Let $k_2 = k_1 + n$ and let

$$S_n = S_n(k_1) = \sum_{k=k_1}^{k_1+n} (F^k - E^k).$$

Then using Lemma 2.15 we see that the operators $F_i(I + S_n)^{-1}$ for $i \in J_k$, $k_1 \leqslant k \leqslant k_1 + n$, and $(I - \sum_{k=k_1}^{k_2} E^k)(I + S_n)^{-1}$ are a disjoint collection of projections. It follows easily that the projections $E_i(I + S_n)^{-1}$, $i \in J_k$, $k < k_1$; $F_i(I + S_n)^{-1}$, $i \in J_k$, $k_1 \leqslant k \leqslant k_1 + n$; and $H_i(I + S_n)^{-1}$, $i \in J_k$, $k > k_1 + n$ are a disjoint collection of projections summing to I. Further, the sum over any subcollection is uniformly bounded with a bound independent of n. That is, the collection of projections define a spectral measure $P^n(\cdot)$ on the integers. To see this we first note that $S_n = O(b2^{k_1[4r-3]/3})$ and thus, assuming k_1 has been chosen so that $O(b2^{k_1[4r-3]/3}) < 1/2$, $(I + S_n)^{-1}$ has a bound independent of n. Clearly any sum of terms $E_i(I + S_n)^{-1}$ or $H_i(I + S_n)^{-1}$ has a bound equal to that of $(I + S_n)^{-1}$. To complete the demonstration of boundedness, it suffices to show that any sum ΣF_i over an index set J which is a subset of $J_{k_1} \cup \cdots \cup J_{k_1+n}$, has a bound independent of n. However

$$\begin{aligned}\sum_{J\cap J_k} F_i &= \sum_{J\cap J_k} \tilde{F}_i(I + F^k - E^k)\\ &= \sum_{J\cap J_k} N_k^{-1}W_k^{-1}H_iW_kN_k(I + F^k - E^k)\\ &= \sum_{J\cap J_k} H_i + O(b2^{k[4r-3]/3})\end{aligned}$$

and thus

$$\begin{aligned}\Big\|\sum_J F_i\Big\| &= \Big\|\sum_J H_i + \sum_{k=k_1}^{k_1+n} O(b2^{k[4r-3]/3})\Big\|\\ &= 1 + O(b2^{k_1[4r-3]/3}) < 3/2.\end{aligned}$$

We want to apply Lemma 2.23 to take a limit of $P^n(\cdot)$ as $n \to \infty$. We have already shown that they are uniformly bounded and need only verify that they form a Cauchy sequence as required in the hypothesis of the lemma. Given $m > n > 0$ and any index set J, let $L_1 = J \cap (J_{k_1+n} \cup \cdots \cup J_{k_1+m})$ and $L_2 = J - L_1$. Since

$$\begin{aligned}\|(I + S_n)^{-1} - (I + S_m)^{-1}\| &= \|(I + S_n)^{-1}(S_m - S_n)(I + S_m)^{-1}\|\\ &= O(b2^{(k_1+n)[4r-3]/3}),\end{aligned}$$

it immediately follows that $\| P^n(L_2) - P^m(L_2)\| = O(b2^{(k_1+n)[4r-3]/3})$. For $i \in L_1$ we know that

$$\sum_{L_1} F_i = \sum_{L_1} H_i + O(b2^{(k_1+n)[4r-3]/3})$$

and a simple estimate shows that $\| P^n(L_1) - P^m(L_1)\|$ is also $O(b2^{(k_1+n)[4r-3]/3})$. The measures P^n then have a limit P which assigns to each integer a rank one projection (or possibly zero for large negative integers). For $i \in J_k$ and $k_0 \leqslant k < k_1$, $\mathscr{R}(P(i)) = \mathscr{R}(E_i)$, while for all other i, $\mathscr{R}(P(i)) = \mathscr{R}(F_i)$. That is, if a finite number of the vectors $\{x_i\}$ are replaced by correspondingly numbered e_i's, the resulting set is an unconditional basis for $\mathfrak{H}$. More precisely, the set

$$x_1, x_2, \ldots, x_{n_{k_1}}$$

is replaced. We want to show that without replacement the set of all $\{x_i\}$ is an unconditional basis. First, the set $\{x_i\}$ is unconditional. The only way it can fail to be, is for some linear combination of

$$x_1, \ldots, x_{n_{k_1}}$$

to be in the closed span $\mathscr{L}$ of the remaining infinite set of eigenvectors. However, from the continuity of $Z(x)$ and Lemma 2.5 we see that for $x \in \mathscr{L}$, $Z(x) < m_{k_1}$, while if x is in the span of

$$x_1, \ldots, x_{n_{k_1}},$$

$Z(x) > m_{k_1}$. Thus $\{x_i\}$ is unconditional. In the quadratic case this also follows from the nonsingularity of $(\lambda_i + \lambda_j)(Bx_i, x_j) + (x_i, x_j)$ (cf. [*16*]).

Having $\{x_i\}$ unconditional, we need only show that its span is $\mathfrak{H}$. Since the set obtained by replacing

$$x_1, \ldots, x_{n_{k_1}}$$

with corresponding vectors e_i is a basis, there are precisely n_{k_1} linearly-independent linear functionals vanishing on $\mathscr{L}$. If the total collection $\{x_i\}$ could be extended to a larger unconditional set, there would be more than n_{k_1} such functionals. Hence $\{x_i\}$ must span $\mathfrak{H}$ and be an unconditional basis.

3. Variational Principles

In this section we show that if (1.1) has a basis of eigenvectors for nonnegative eigenvalues, then the eigenvalues are characterized by variational principles. If no assumption is made about the eigenvectors, then a certain number of the eigenvalues can still be so characterized. By a basis we mean a set $\{h_i\}$ such that each x can be uniquely represented as $\lim_{N\to\infty} \sum_{i=1}^{N} a_i h_i$. As in [*16*] we let $\mathscr{E}^n$ denote a generic n dimensional subspace of $\mathfrak{H}$.

THEOREM 3.1. *Suppose that the eigenvectors of $A - B(\lambda)$, corresponding to nonnegative eigenvalues, form a basis for $\mathfrak{H}$. Then*

$$\lambda_n = \min_{\mathscr{E}^{n-1}} \max_{x \perp \mathscr{E}^{n-1}} Z(x), \tag{3.1}$$

the value λ_n being assumed for an eigenvector x_n.

Proof. If the eigenvectors $\{x_i\}$ form a basis, then there exists a dual set of vectors x_i^* satisfying $(x_i^*, x_j) = \delta_{ij}$, the Kronecker delta. Suppose $\mathscr{E}^{n-1}$ is the linear span of $x_1^*, x_2^*, \ldots, x_{n-1}^*$. Then x_n is orthogonal to $\mathscr{E}^{n-1}$ and $Z(x_n) = \lambda_n$. Moreover, any vector orthogonal to $\mathscr{E}^{n-1}$ can be written as $x_0 + \lim_{m\to\infty} \sum_{i=n}^{m} a_i x_i$, where x_0 is a null-vector of A. By Lemma 2.5, $Z(x_0 + \sum_{i=n}^{m} a_i x_i) \leqslant \lambda_n$ which, since Z is continuous, means that for $x \perp \mathscr{E}^{n-1}$, $Z(x) \leqslant \lambda_n$. For an arbitrary $\mathscr{E}^{n-1}$, we can always find a linear combination of $x_1, x_2, \ldots, x_n$ perpendicular to it, and conclude that for $x \perp \mathscr{E}^{n-1}$, $\max Z(x) \geqslant \lambda_n$, yielding (3.1). We leave it to the reader to show that $\max Z(x)$ is attained.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1

$$\lambda_n = \max_{\mathscr{E}^n} \min_{x\in\mathscr{E}^n} Z(x). \tag{3.2}$$

Proof. Clearly the minimum of $Z(x)$ for x in span of $x_1, x_2, \ldots, x_n$, is λ_n. For any subspace $\mathscr{E}^n$ there is always an $x \in \mathscr{E}^n$ satisfying $(x_i^*, x) = 0$ for $i = 1, 2, \ldots, n = 1$, which means that for $x \in \mathscr{E}^{n-1}$, $\min Z(x) \leqslant \lambda_n$.

Theorems 3.1 and 3.2 are obviously applicable to $A - B(\lambda)$ if $A \in C_r$ with $r < 1/2$ or if $A \in C_r$ with $r < 2/3$ and $B(\lambda) \in C_2$. However, without these conditions and without any assumption on eigenvectors, we can still obtain some of the eigenvalues by variational principles. For that we need the following lemma:

LEMMA 3.3. *Let $C = \{c_{ij}\}$ be a nonnegative complex n by n matrix and let* diag $C = \{\delta_{ij}c_{ij}\}$. *Then for any complex n-vector x,*

$$\langle Cx, x\rangle \leqslant n\langle \operatorname{diag} Cx, x\rangle, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product.

Proof. Since the rank one perpendicular projections and 0 generate the cone of nonnegative matrices under convex combinations and positive multiples and since the inequality (3.3) is preserved under such operations, it suffices to prove it for rank one orthogonal projections. Such projections have the form U^*EU where E corresponds to the matrix $\{\delta_{1j}\delta_{ij}\}$ and U is unitary. For U^*EU, (3.3) takes the form

$$\left|\sum_{j=1}^{n} U_{1j}\bar{x}_j\right|^2 \leqslant n \sum_{j=1}^{n} |U_{1j}|^2 |x_j|^2$$

for $x = (x_1, x_2, \ldots, x_n)$ and follows from Schwarz's inequality.

THEOREM 3.4. *For $n \geqslant 2$ and $k \geqslant 3$ let*

$$C(n, k) = \begin{cases} n(2k-5) & \text{for } k \text{ odd} \\ n(2k-4) & \text{for } k \text{ even.} \end{cases}$$

Then if $\sum_{k=3}^{N} C(n, k) \| B_k \| < 1$, the first $n+1$ eigenvalues of $A - B(\lambda)$ can be characterized successively by

$$\lambda_m = \max_{[x, x_j] = 0} Z(x). \tag{3.4}$$

$$j = 1, 2, \ldots, m-1,$$

the maximum being attained for an eigenvector x_m, linearly independent of $x_1, x_2, \ldots, x_{m-1}$, corresponding to λ_m. With no assumptions on the operators B_k, the first two eigenvalues can be so characterized.

Proof. The proof proceeds exactly like that of Theorem 4.1 of [*16*] with inequality (2.1) replacing compactness of $B_1(\lambda)$, provided that the analog of the matrix M, in this case

$$M \equiv M(\lambda, m) \equiv (\Delta(\lambda, Z(x_i))\, x_i, x_j), \qquad (0 \leqslant \lambda \leqslant 1),$$

is nonsingular for $1 \leqslant i, j \leqslant k$ and each $k \leqslant m$. Thus the theorem is really a statement about M and could be phrased as such. Note that the eigenvectors, still denoted by x_i, may no longer be orthogonal in a single eigenspace.

The matrix

$$M \equiv \sum_{k=2}^{N} \sum_{s=0}^{k} \lambda^{k-1-s} \lambda_j^{s} (B_k x_i, x_j) + (x_i, x_j), \qquad 1 \leqslant i, j \leqslant m,$$

cannot be singular if its real part, $\frac{1}{2}(M + M^*)$, is strictly positive. We have

$$\begin{aligned} M + M^* &= \sum_{k=2}^{N} \sum_{s=0}^{k-1} \lambda^{k-1-s} (\lambda_i^{s} + \lambda_j^{s})(B_k x_i, x_j) + 2(x_i, x_j) \\ &= \sum_{k=2}^{N} \Bigg[\sum_{m=0}^{k-1} \lambda_i^{k-1-m} \lambda_j^{m} (B_k x_i, x_j) - \sum_{m=1}^{k-2} \lambda_i^{k-1-m} \lambda_j^{m} (B_k x_i, x_j) \\ &\qquad + \sum_{s=0}^{k-2} \lambda^{k-1-s} (\lambda_i^{s} + \lambda_j^{s})(B_k x_i, x_j) \Bigg] + 2(x_i, x_j) \\ &\geqslant \sum_{k=3}^{N} \Bigg[k \delta_{ij} \lambda_i^{k-1} (B_k x_i, x_i) - \sum_{m=1}^{k-2} \lambda_i^{k-1-m} \lambda_j^{m} (B_k x_i, x_j) \\ &\qquad + \sum_{s=0}^{k-2} \lambda^{k-1-s} (\lambda_i^{s} + \lambda_j^{s})(B_k x_i, x_j) \Bigg] + (1 + \delta_{ij})(x_i, x_j) \end{aligned} \tag{3.5}$$

using the fact that $[x_i, x_j] = 0$ for $i \neq j$, and the nonnegativity of B_2. We can assume inductively that $x_1, x_2, \ldots, x_m$ are linearly independent and normalized. We denote the matrix $(B_k x_i, x_j)$ by $\hat{B}_k$; $(\delta_{ij}\lambda_j)$, by Λ; (x_i, x_j) by X, and δ_{ij} by I. With $\langle \cdot, \cdot \rangle$ still denoting the inner product in complex m-space and v a complex m-vector, it suffices to show, referring to (3.5) and the hypotheses of the theorem, that

$$k\langle \operatorname{diag}(\Lambda^{k-1}\hat{B}_k)\, v, v\rangle - \sum_{p=1}^{k-2} \langle \Lambda^{k-1-p}\hat{B}_k\Lambda^p v, v\rangle$$

$$+ \sum_{s=0}^{k-2} \lambda^{k-1-s}\langle (\Lambda^s\hat{B}_k + \hat{B}_k\Lambda^s)\, v, v\rangle + C(n, k)\| B_k \|\langle (I + X)\, v, v\rangle \tag{3.6}$$

is positive for each $k \geqslant 3$.

Consider the term $\lambda^{k-1-s}\langle \Lambda\hat{B}_k + \hat{B}_k\Lambda)v, v\rangle$ for $s \geqslant \frac{1}{2}(k-1)$. Using the Schwarz inequality with respect to $\hat{B}_k$ which is nonnegative, the inequality $2ab \leqslant \eta a^2 + (1/\eta)b^2$, and Lemma 3.3 we see that

$$\begin{aligned} |\lambda^{k-1-s}\langle (\Lambda^s\hat{B}_k + \hat{B}_k\Lambda^s)\, v, v\rangle| &\leqslant 2\lambda^{k-1-s}\, |\langle \hat{B}_k v, \Lambda^s v\rangle| \\ &\leqslant 2\lambda^{k-1-s}\langle \hat{B}_k v, v\rangle^{1/2} \cdot \langle \hat{B}_k\Lambda^s, \Lambda^s\rangle^{1/2} \\ &\leqslant \eta\lambda^{2(k-1-s)}\langle \hat{B}_k v, v\rangle + \frac{m}{\eta}\langle \operatorname{diag} \Lambda^{2s}\hat{B}_k v, v\rangle. \end{aligned} \tag{3.7}$$

Suppose k is odd. Then using (3.7), one sees that (3.6) is no smaller than

$$k\langle \operatorname{diag}(\Lambda^{k-1}\hat{B}_k)\, v, v\rangle - \sum_{s=(k-1)/2}^{k-2} \frac{m}{\eta}\langle \operatorname{diag}(\Lambda^{2s}\hat{B}_k)\, v, v\rangle$$

$$- \eta \sum_{s=(k+1)/2}^{k-2} \lambda^{2(k-1-s)}\langle \hat{B}_k v, v\rangle + (2-\eta)\, \lambda^{k-1}\langle \hat{B}_k v, v\rangle$$

$$+ \sum_{s=1}^{1/2(k-3)} \lambda^{k-1-s}\langle (\Lambda^s\hat{B}_k + \hat{B}_k\Lambda^s)\, v, v\rangle - \sum_{p=1}^{k-2} \langle \Lambda^{k-1-p}\hat{B}_k\Lambda^p v, v\rangle$$

$$+ n(2k-5)\| B_k \|\langle (I + X)\, v, v\rangle. \tag{3.8}$$

An upper bound for $\langle \hat{B}_k v, v\rangle$ is $m\| B_k \|\langle v, v\rangle$. Then since all eigenvalues λ_i lie in $[0, 1]$ and $0 \leqslant \lambda \leqslant 1$, (3.8) is bounded below by

$$\left(k - \frac{m}{\eta}\left(\frac{k-1}{2}\right)\right) \langle \operatorname{diag} \Lambda^{k-1}\hat{B}_k v, v\rangle - \eta\left(\frac{k-3}{2}\right) \lambda^{k-1}\langle \hat{B}_k v, v\rangle$$
$$+ (2-\eta)\, \lambda^{k-1}\langle \hat{B}_k v, v\rangle - \tfrac{1}{2}(k-3)\cdot 2\cdot m \| B_k \|\langle v, v\rangle$$
$$- (k-2)\, m \| B_k \|\langle v, v\rangle + n(2k-5)\| B_k \|\langle (I+X)\, v, v\rangle$$
$$\geqslant \left(k - \frac{n}{\eta}\left(\frac{k-1}{2}\right)\right) \langle \operatorname{diag}(\Lambda^{k-1}\hat{B}_k)\, v, v\rangle - n(2k-5)\| B_k \|\langle v, v\rangle$$
$$+ \left(2 - \eta\left(\frac{k-1}{2}\right)\right) \langle \hat{B}_k v, v\rangle + n(2k-5)\| B_k \|\langle (I+X)\, v, v\rangle. \quad (3.9)$$

If we let $w = \sum_{i=1}^{m} v_i x_i$, then $\langle \hat{B}_k v, v\rangle = (B_k w, w)$ and $\langle Xv, v\rangle = (w, w)$. Setting

$$\eta = \frac{n}{k}\left(\frac{k-1}{2}\right) \quad \text{and noting that} \quad \frac{n}{k}\left(\frac{k-1}{2}\right)^2 - 2 \leqslant n(2k-5),$$

we see that (3.9) is positive. Similar estimates hold if k is even, though in that case the term $2\lambda^{k-1}\langle \hat{B}_k v, v\rangle$ is not readily useable in the estimate. This completes the proof except for the last assertion of the theorem and that follows from the positivity of $M(\lambda, 1)$.

Remarks. One can improve the estimates in the preceding proof in some cases. For example, if $k = 3$ and $n = 2$, the term $-\langle \Lambda \hat{B}_3 \Lambda v, v\rangle$ in (3.8) can be bounded below by $-2\langle \operatorname{diag} \Lambda^2 \hat{B}_3 v, v\rangle$ which, with $\eta = 2$, obviates the need for $2\| B_k \|\langle (I+X)v, v\rangle$ in making (3.8) positive. As a result, for any cubic $B(\lambda)$, the first three eigenvalues can be obtained from (3.4). Further, in an application where one had specific information about $B(\lambda)$, better estimates of M might be possible.

THEOREM 3.5. *Under the hypotheses of Theorem* 3.4

$$\lambda_m = \min_{\mathscr{E}_{n-1}} \max_{x \perp \mathscr{E}_{n-1}} Z(x)$$

and

$$\lambda_m = \max_{\mathscr{E}_n} \min_{x \in \mathscr{E}_n} Z(x),$$

the extreme values being assumed for an eigenvector x_m.

Proof. Compare with [*16*].

As corollaries of Theorems 3.1 and 3.4, we have the following two results (cf. [*16*]):

COROLLARY 3.6. *Let P be the orthogonal projection of $\mathfrak{H}$ onto a subspace $\mathcal{M}$ of dimension d. Suppose that the first d eigenvalues $\lambda_1' \geqslant \lambda_2' \geqslant \cdots \geqslant \lambda_d'$ of $P(A) - B(\lambda))P$ in $\mathcal{M}$ and the first d of $A - B(\lambda)$ in $\mathfrak{H}$ are characterized by the minimax principle. Then $\lambda_i' \leqslant \lambda_i$ for $i \leqslant d$.*

COROLLARY 3.7. *Let $C(\lambda)$ be any polynomial with nonnegative coefficients satisfying $C(\lambda) \leqslant B(\lambda)$ for $0 \leqslant \lambda \leqslant 1$, and denote the eigenvalues of $A - C(\lambda)$ by μ_1. Then $\lambda_i \leqslant \mu_i$ when both are characterized by the minimax principle.*

To conclude this section we briefly describe a method for generating polynomials $C(\lambda) \leqslant B(\lambda)$. It suffices to subordinate the coefficients and thus we consider a single nonnegative operator B. The method was introduced by N. Aronszajn in [*22*] and has been applied to linear problems by N. Bazley and D. Fox (see, for example, [*23*]).

For simplicity we assume B has a trivial nullspace, i.e. $B > 0$. Then with the inner product $\langle f, g\rangle = (B^{1/2}f, B^{1/2}g)$ and norm $\|f\|_1 = \langle f, f\rangle^{1/2}$, $\mathfrak{H}$ is a pre-Hilbert space with a completion $\mathfrak{H}_1 \supset \mathfrak{H}$. One can show that $\mathfrak{H}$ is dense in $\mathfrak{H}_1$ in the norm $\|\ \|_1$. Thus a set of vectors $\{g_i\}$ which span $\mathfrak{H}$ will also span $\mathfrak{H}_1$. Let $\{h_j\}$ be a set of vectors obtained by orthonormalizing the set $\{g_i\}$ with respect to the inner product $\langle\cdot, \cdot\rangle$. Thus Q_l defined by $Q_l f = \sum_{j=1}^{l} \langle f, h_j\rangle h_j$ is a perpendicular projection in $\mathfrak{H}_1$. Since $(BQ_l f, f) = \langle Q_l f, f\rangle$ for f in $\mathfrak{H}$, one sees that

$$0 \leqslant BQ_l \leqslant BQ_{l+1} \leqslant B$$

in $\mathfrak{H}$.

Note added in proof. Professor T. Kato has kindly pointed out that Lemma 2.13 can be obtained using Theorem 6.35 on page 58 of his book, "Perturbation Theory for Linear Operators", Springer, New York, 1966.

REFERENCES

1. MÜLLER, P. H., Eine neue Methode zur Behandlung nichtlinearer Eigenvertaufgaben. *Math. Z.* **70** (1959), 381–406.
2. LANCASTER, P., "Lambda-Matrices and Vibrating Systems." Pergamon, Oxford, England 1966.
3. ANSELONE, P., AND RALL, L., "Newton's Method for Characteristic Vector-Value Problems." *Tech. Summary Rept.* #492, Mathematics Research Center, U.S. Army, Madison, Wisconsin.
4. BARSTON, E. M., Electrostatic resonance oscillations of a nonuniform hot plasma in an external field, *Phys. Rev.* **139**, (1965), A394–A400.

5. Chandrasekhar, S., Hydrodynamic and Hydromagnetic Stability. Clarendon, Oxford, England 1961.
6. Doležal, V., Location of eigenfrequencies of electric networks. *J. Soc. Indust. Appl. Math.* **12** (1964), 526–538.
7. Förster, K.-H., Über linear, abgeschlossene Operatoren, die analytisch von einem Parameter abhangen. *Math. Z.* **95** (1967), 251–258.
8. Hadeler, K. P., Eigenverte von operatorpolynomen. *Arch. Rational Mech. Anal.* **20** (1965), 72–80.
9. Hadeler, K. P., Über operatorgleichungen mit nicht linear auftretendem parameter, *ZAMM* **47** (1967), 91–96.
10. Harazov, D. F., Spectral theory of completely continuous linear operators depending quadratically on a parameter. *Akad. Nauk Gruzin SSR Trudy Tbiliss. Mat. Inst. Razmadze* **26** (1959), 171–187.
11. Harazov, D. F., Spectral theory of some linear operators depending meromorphically on a parameter. *Studia Math.* **20** (1961), 19–45.
12. Harazov, D. F., On the spectrum of completely continuous operators depending analytically on a parameter, in a topological linear space. *Acta Sci. Math. (Szeged)* **23** (1962), 38–45.
13. Rogers, E. H., A minimax theory for over damped systems. *Arch. Rational Mech. Anal.* **16** (1964), 89–96.
14. Shinbrot, M., Note on a nonlinear eigenvalue problem. *Proc. Am. Math. Soc.* **14** (1963), 552–558.
15. Shinbrot, M., A nonlinear eigenvalue problem, II. *Arch. Ration. Mech. Anal.* **15** (1964), 368–376.
16. Turner, R. E. L., Some variational principles for a nonlinear eigenvalue problem. *J. Math. Anal. Appl.* **17** (1967), 151–160.
17. Schwartz, J., Perturbations of spectral operators, and applications. *Pacific J. Math.* **4** (1954), 415–458.
18. Riesz, F., and Sz.-Nagy, B., "Functional Analysis." Ungar, New York, 1955.
19. Dunford, N. and Schwartz, J., Linear operators, vol. II, Spectral Theory. Wiley, New York, 1963.
20. Wermer, J., Commuting spectral measures on Hilbert space. *Pacific J. Math.* **4** (1954), 355–361.
21. Lorch, E. R., Bicontinuous linear transformations in certain vector spaces. *Bull. Am. Math. Soc.* **45** (1939), 564–569.
22. Aronszajn, N., Approximation methods for eigenvalues of completely continuous symmetric operators. *Proc. Symp. Spectral Theory and Differential Problems*, Stillwater, Oklahoma, 1951.
23. Bazley, N. W., and Fox, D. W., Lower bounds for eigenvalues of Schrödinger's equation. *Phys. Rev.* **124** (1961), 483–492.